

SOME IDENTITIES INVOLVING ASSOCIATED SEQUENCES OF SPECIAL POLYNOMIALS

TAEKYUN KIM¹ AND DAE SAN KIM²

ABSTRACT. In this paper, we study some properties of associated sequences of special polynomials. From the properties of associated sequences of polynomials, we derive some interesting identities of special polynomials.

1. INTRODUCTION

For $r \in \mathbb{R}$, the *Bernoulli polynomials* of order r are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{R}), \quad (\text{see [12,13,14,18,21]}). \quad (1.1)$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the n -th Bernoulli numbers of order r . It is also well known that the *Euler polynomials* of order r are defined by the generating function to be

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{R}), \quad (\text{see [9,10,11,19,20]}). \quad (1.2)$$

Let $x = 0$. Then $E_n^{(r)}(0) = E_n^{(r)}$ are called the n -th *Euler numbers* of order r .

Let \mathcal{F} be the set of all formal power series in the variable t over \mathbb{C} with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbb{C} \right\}. \quad (1.3)$$

Let \mathbb{P} be the algebra of polynomials in the variable x over \mathbb{C} and \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . The action of the linear functional L on a polynomial $p(x)$ is defined by $\langle L | p(x) \rangle$ and the vector space structure on \mathbb{P}^* is derived by $\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle$, $\langle cL | p(x) \rangle = c \langle L | p(x) \rangle$, where c is a complex constant.

For $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$, we define a linear functional on \mathbb{P} by setting

$$\langle f(t) | x^n \rangle = a_n, \quad (n \geq 0), \quad (\text{see [3,8,17]}). \quad (1.4)$$

By (1.3) and (1.4), we get

$$\langle t^k | x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [4,5,7,10,17,18]}), \quad (1.5)$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k$. Then, by (1.5), we get $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$. So, we see that $f_L(xt) = L$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^*

1991 *Mathematics Subject Classification.* 05A10, 05A19.

Key words and phrases. Bernoulli polynomial, Euler polynomial, Abel polynomial.

onto \mathcal{F} . Henceforth, \mathcal{F} is thought of as both a formal power series and a linear functional (see [4, 6, 8, 17, 18]). We call \mathcal{F} the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [4, 10, 17]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a *delta series* and if $o(f(t)) = 0$, then $f(t)$ is called an *invertible series*. Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k | S_n(x) \rangle = n! \delta_{n,k}$ where $n, k \geq 0$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the *associated sequence* for $f(t)$. By (1.5), we see that $\langle e^{yt} | p(x) \rangle = p(y)$.

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k | p(x) \rangle}{k!} x^k, \quad (\text{see [10,17]}), \quad (1.6)$$

and

$$\langle f_1(t)f_2(t) \cdots f_m(t) | x^n \rangle = \sum_{i_1 + \cdots + i_m = n} \binom{n}{i_1, \dots, i_m} \langle f_1(t) | x^{i_1} \rangle \cdots \langle f_m(t) | x^{i_m} \rangle, \quad (1.7)$$

where $f_1(t), f_2(t), \dots, f_m(t) \in \mathcal{F}$, (see [4, 10, 17]). From (1.6), we have

$$p^{(k)}(0) = \langle t^k | p(x) \rangle, \quad \langle 1 | p^{(k)}(x) \rangle = p^{(k)}(0). \quad (1.8)$$

Thus, by (1.8), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0), \quad (\text{see [17]}). \quad (1.9)$$

For $S_n(x) \sim (g(t), f(t))$, we have the following equations:

$$S_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(y) S_{n-k}(x), \quad \text{where } p_k(y) = g(t) S_k(y), \quad (1.10)$$

and

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C} \quad (\text{see [10,15,16,17,18]}), \quad (1.11)$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula for associated sequence implies that, for $n \in \mathbb{N}$,

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x), \quad (\text{see [11,17,22]}). \quad (1.12)$$

Now we introduce several important sequences which are used to derive our results in this paper (see [10, 11, 17]):

(The Poisson-Charlier sequences)

$$C_n(x; a) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim (e^{a(e^t-1)}, a(e^t-1)),$$

where $a \neq 0$, $(x)_n = x(x-1) \cdots (x-n+1)$,

$$\sum_{k=0}^n C_n(k; a) \frac{t^k}{k!} e^{-t} = \left(\frac{t-a}{a} \right)^n, \quad (a \neq 0), \quad n \in \mathbb{N} \cup \{0\}, \quad (1.13)$$

(The Abel sequences)

$$A_n(x; b) = x(x-bn)^{n-1} \sim (1, te^{bt}), \quad (b \neq 0), \quad (1.14)$$

(The Mittag-Leffler sequences)

$$M_n(x) = \sum_{k=0}^n \binom{n}{k} (n-1)_{n-k} 2^k (x)_k \sim \left(1, \frac{e^t - 1}{e^t + 1} \right), \quad (1.15)$$

(The exponential sequences)

$$\phi_n(x) = \sum_{k=0}^n S_2(n, k) x^k \sim (1, \log(1+t)), \quad (1.16)$$

and

(The Laguerre sequences)

$$L_n(x) = \sum_{k=1}^n \binom{n-1}{k-1} \frac{n!}{k!} (-x)^k \sim \left(1, \frac{t}{t-1} \right). \quad (1.17)$$

In this paper, we study some properties of associated sequences of special polynomials. From the properties of associated sequences of special polynomials, we derive some interesting identities involving associated sequences of special polynomials.

2. ASSOCIATED SEQUENCES OF SPECIAL POLYNOMIALS.

As is well known, the Bessel differential equation is given by

$$x^2 y'' + 2(x+1)y' + n(n+1)y = 0, \quad (\text{see [1,2]}). \quad (2.1)$$

From (2.1), we have the solution of (2.1) as follows:

$$y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2} \right)^k, \quad (\text{see [1,2]}). \quad (2.2)$$

Let us consider the following associated sequences:

$$p_n(x) \sim \left(1, t - \frac{t^2}{2} \right), \quad x^n \sim (1, t), \quad (\text{see [1,2,10,17]}). \quad (2.3)$$

From (1.12) and (2.3), for $n \in \mathbb{N}$, we have

$$\begin{aligned}
p_n(x) &= x \left(\frac{t}{t - \frac{t^2}{2}} \right)^n x^{-1} x^n = x \left(1 - \frac{t}{2} \right)^{-n} x^{n-1} \\
&= x \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k \left(\frac{t}{2} \right)^k x^{n-1} \\
&= x \sum_{k=0}^{n-1} \binom{n+k-1}{k} \left(\frac{1}{2} \right)^k (n-1)_k x^{n-1-k} \\
&= \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-1-k)!} \left(\frac{1}{2} \right)^k x^{n-k} \\
&= \sum_{k=1}^n \frac{(2n-k-1)!}{(n-k)!(k-1)!} \left(\frac{1}{2} \right)^{n-k} x^k.
\end{aligned} \tag{2.4}$$

By (2.2) and (2.4), we get

$$p_n(x) = x^n y_{n-1} \left(\frac{1}{x} \right) \sim \left(1, t - \frac{t^2}{2} \right). \tag{2.5}$$

From (1.11) and (2.5), we can derive the following generating function of $p_n(x)$:

$$\sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!} = e^{x(1-(1-2t)^{\frac{1}{2}})}, \tag{2.6}$$

and, by (1.10), we get

$$(x+y)^n y_{n-1} \left(\frac{1}{x+y} \right) = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} y_{k-1} \left(\frac{1}{x} \right) y_{n-k-1} \left(\frac{1}{y} \right). \tag{2.7}$$

By (1.12) and (2.3), we get

$$x^n = x \left(\frac{t - \frac{t^2}{2}}{t} \right)^n x^{-1} p_n(x) = x \left(\frac{t-2}{-2} \right)^n x^{-1} p_n(x). \tag{2.8}$$

Thus, by (1.13), (2.4) and (2.8), we get

$$\begin{aligned}
(-1)^n x^{n-1} &= \left(\frac{t-2}{2} \right)^2 x^{-1} p_n(x) = \sum_{k=0}^{\infty} C_n(k; 2) \frac{t^k}{k!} e^{-t} (x^{-1} p_n(x)) \\
&= \sum_{k=0}^{n-1} C_n(k; 2) \frac{t^k}{k!} (x-1)^{-1} p_n(x-1) \\
&= \sum_{k=0}^{n-1} C_n(k; 2) \frac{t^k}{k!} \sum_{l=1}^n \frac{(2n-l-1)!}{(l-1)!(n-l)!} \left(\frac{1}{2} \right)^{n-l} (x-1)^{l-1} \\
&= \sum_{k=0}^{n-1} C_n(k; 2) \sum_{l=k+1}^n \frac{(2n-l-1)!}{(l-1)!(n-l)!} \binom{l-1}{k} \left(\frac{1}{2} \right)^{n-l} (x-1)^{l-1-k} \\
&= \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} C_n(k; 2) \binom{m+k}{k} \frac{(2n-m-k-2)!}{(m+k)!(n-m-k-1)!} \left(\frac{1}{2} \right)^{n-m-k-1} (x-1)^m.
\end{aligned} \tag{2.9}$$

From (2.8), we have

$$\begin{aligned}
x^{n-1} &= \left(1 - \frac{t}{2}\right)^n x^{-1} p_n(x) = \sum_{k=0}^n \binom{n}{k} \left(-\frac{t}{2}\right)^k x^{-1} p_n(x) \\
&= \sum_{k=0}^{n-1} \sum_{l=k+1}^n \binom{n}{k} (l-1)_k \frac{(2n-l-1)!}{(l-1)!(n-l)!} (-1)^k \left(\frac{1}{2}\right)^{n+k-l} x^{l-1-k} \\
&= \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} \binom{n}{k} (m+k)_k \frac{(2n-m-k-2)!}{(m+k)!(n-m-1-k)!} \left(\frac{1}{2}\right)^{n-m-1} x^m \\
&= \sum_{m=0}^{n-1} \left\{ \sum_{k=0}^{n-m-1} (-1)^k \left(\frac{1}{2}\right)^{n-m-1} \binom{n}{k} \frac{(2n-m-k-2)!}{m!(n-m-k-1)!} \right\} x^m.
\end{aligned} \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, we have*

$$(-1)^n x^{n-1} = \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} C_n(k; 2) \binom{m+k}{k} \frac{(2n-m-k-2)!}{(m+k)!(n-m-k-1)!} \left(\frac{1}{2}\right)^{n-m-k-1} (x-1)^m.$$

Moreover,

$$\sum_{k=0}^{n-m-1} (-1)^k \left(\frac{1}{2}\right)^{n-m-1} \binom{n}{k} \frac{(2n-m-k-2)!}{m!(n-m-k-1)!} = 0,$$

where $0 \leq m \leq n-2$.

Let us consider the following associated sequences:

$$p_n(x) \sim \left(1, te^{c(e^t-1)}\right), \quad c \neq 0, \quad A_n(x; b) = x(x-bn)^{n-1} \sim (1, te^{bt}), \quad b \neq 0. \tag{2.11}$$

By (1.12) and (2.11), we get

$$p_n(x) = x \left(\frac{t}{te^{c(e^t-1)}} \right)^n x^{-1} x^n = x \sum_{k=0}^{\infty} \frac{(-nc)^k}{k!} (e^t - 1)^k x^{n-1}. \tag{2.12}$$

We recall that Newton's difference operator Δ is defined by $\Delta f(x) = f(x+1) - f(x)$. For $n \in \mathbb{N}$, we easily see that

$$\Delta^n p(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} p(x+k). \tag{2.13}$$

By (2.13), we get

$$(e^t - 1)^k p(x) = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{lt} p(x) = \sum_{l=0}^k \binom{k}{l} (-1)^{l-k} p(x+l) = \Delta^k p(x). \tag{2.14}$$

In particular, if we take $p(x) = x^{n-1}$, then we have

$$(e^t - 1)^k x^{n-1} = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (x+j)^{n-1}. \tag{2.15}$$

From (2.12) and (2.15), we have

$$p_n(x) = x \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{(-1)^j (nc)^k}{k!} \binom{k}{j} (x+j)^{n-1} \sim \left(1, te^{c(e^t-1)}\right), \quad c \neq 0. \quad (2.16)$$

Therefore, by (2.16), we obtain the following lemma.

Lemma 2.2. *For $c \neq 0$ and $n \in \mathbb{N}$, let $p_n(x) \sim \left(1, te^{c(e^t-1)}\right)$. Then we have*

$$p_n(x) = x \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{(-1)^j (nc)^k}{k!} \binom{k}{j} (x+j)^{n-1}.$$

From the definition of Abel sequences and (2.11), we note that

$$\begin{aligned} A_n(x; b) &= x(x - bn)^{n-1} = x \left(\frac{te^{c(e^t-1)}}{te^{bt}} \right)^n x^{-1} p_n(x) \\ &= x \left(e^{-bt-c(1-e^t)} \right)^n x^{-1} p_n(x) = x \left(\sum_{l=0}^{\infty} \frac{a_l^{(-b)}(-c)}{l!} t^l \right)^n x^{-1} p_n(x), \end{aligned} \quad (2.17)$$

where $a_n^{(\beta)}(x) \sim ((1-t)^{-\beta}, \log(1-t))$ is the *actuarial polynomial* with the generating function given by

$$\sum_{l=0}^{\infty} \frac{a_l^{(\beta)}(x)}{l!} t^l = e^{\beta t + x(1-e^t)}.$$

By Lemma 2.2 and (2.17), we get

$$\begin{aligned} &A_n(x; b) \\ &= x \left\{ \sum_{m=0}^{\infty} \sum_{l_1+\dots+l_n=m} \binom{m}{l_1, \dots, l_n} a_{l_1}^{(-b)}(-c) \cdots a_{l_n}^{(-b)}(-c) \frac{t^m}{m!} \right\} \\ &\quad \times \left\{ \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{(-1)^j (nc)^k}{k!} \binom{k}{j} (x+j)^{n-1} \right\} \\ &= x \sum_{m=0}^{n-1} \sum_{l_1+\dots+l_n=m} \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{n-1}{m} \binom{m}{l_1, \dots, l_n} \binom{k}{j} \left(\prod_{j=1}^n a_{l_j}^{(-b)}(-c) \right) \left(\frac{(-1)^j (nc)^k}{k!} \right) (x+j)^{n-1-m}. \end{aligned} \quad (2.18)$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$, $b \neq 0$, $c \neq 0$, we have*

$$\begin{aligned} &A_n(x; b) \\ &= x \sum_{m=0}^{n-1} \sum_{l_1+\dots+l_n=m} \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{n-1}{m} \binom{m}{l_1, \dots, l_n} \binom{k}{j} \left(\prod_{j=1}^n a_{l_j}^{(-b)}(-c) \right) \left(\frac{(-1)^j (nc)^k}{k!} \right) (x+j)^{n-1-m}. \end{aligned}$$

For (2.17), we note that

$$\begin{aligned} A_n(x; b) &= x \left(e^{c(e^t-1)} \right)^n e^{-nbt} x^{-1} p_n(x) \\ &= x \left(e^{c(e^t-1)} \right)^n (x - nb)^{-1} p_n(x - nb). \end{aligned} \quad (2.19)$$

By (1.16) and Lemma 2.2, we easily see that the generating function of exponential sequences is given by

$$\sum_{k=0}^{\infty} \phi_k(x) \frac{t^k}{k!} = e^{x(e^t-1)}. \quad (2.20)$$

From (2.19) and (2.20), we have

$$\begin{aligned} A_n(x; b) &= x \left\{ \sum_{m=0}^{\infty} \sum_{l_1+\dots+l_n=m} \binom{m}{l_1, \dots, l_n} \left(\prod_{j=1}^n \phi_{l_j}(c) \right) \frac{t^m}{m!} \right\} (x - nb)^{-1} p_n(x - nb) \\ &= x \sum_{m=0}^{\infty} \sum_{l_1+\dots+l_n=m} \binom{m}{l_1, \dots, l_n} \left(\prod_{j=1}^n \phi_{l_j}(c) \right) \frac{t^m}{m!} \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{(-1)^j (nc)^k}{k!} \binom{k}{j} (x - nb + j)^{n-1} \\ &= x \sum_{m=0}^{n-1} \sum_{l_1+\dots+l_n=m} \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{n-1}{m} \binom{m}{l_1, \dots, l_n} \binom{k}{j} \left(\prod_{j=1}^n \phi_{l_j}(c) \right) \frac{(-1)^j (nc)^k}{k!} (x - nb + j)^{n-m-1}. \end{aligned} \quad (2.21)$$

Therefore, by (2.21), we obtain the following corollary.

Corollary 2.4. *For $n \geq 1$, $b \neq 0$, $c \neq 0$, we have*

$$A_n(x; b) = x \sum_{m=0}^{n-1} \sum_{l_1+\dots+l_n=m} \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{n-1}{m} \binom{m}{l_1, \dots, l_n} \binom{k}{j} \left(\prod_{j=1}^n \phi_{l_j}(c) \right) \frac{(-1)^j (nc)^k}{k!} (x - nb + j)^{n-m-1}.$$

Note that $x^n \sim (1, t)$. By (1.12), (1.13) and (1.17), we get

$$\begin{aligned} L_n(x) &= x \left(\frac{t}{\frac{t}{t-1}} \right)^n x^{-1} x^n = x(t-1)^n x^{n-1} \\ &= x \left(\sum_{k=0}^{n-1} C_n(k; 1) \frac{t^k}{k!} e^{-t} \right) x^{n-1} = x \sum_{k=0}^{n-1} C_n(k; 1) \frac{t^k}{k!} (x-1)^{n-1} \\ &= x \sum_{k=0}^{n-1} C_n(k; 1) \binom{n-1}{k} (x-1)^{n-1-k} = x \sum_{k=0}^{n-1} \binom{n-1}{k} C_n(n-1-k; 1) (x-1)^k. \end{aligned} \quad (2.22)$$

Therefore, by (2.22), we obtain the following theorem.

Theorem 2.5. *For $n \geq 1$, we have*

$$L_n(x) = x \sum_{k=0}^{n-1} \binom{n-1}{k} C_n(n-1-k; 1) (x-1)^k.$$

Mott considered the associated sequences for $f(t) = \frac{-2t}{1-t^2}$. That is, the Mott sequence is given by

$$S_n(x) \sim \left(1, \frac{-2t}{1-t^2} \right). \quad (2.23)$$

From (2.23), we note that the generating function of Mott sequences is given by

$$\sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!} = \exp \left(x \left(\frac{1 - \sqrt{1+t^2}}{t} \right) \right).$$

By (1.12), (1.17) and (2.23), we get

$$\begin{aligned} S_n(x) &= x \left(\frac{\frac{t}{t+1}}{\frac{-2t}{1-t^2}} \right)^n x^{-1} L_n(-x) = x \left(\frac{t-1}{2} \right)^n x^{-1} L_n(-x) \\ &= 2^{-n} x (t-1)^n x^{-1} L_n(-x) = 2^{-n} x \left(\sum_{k=0}^{n-1} C_n(k; 1) \frac{t^k}{k!} e^{-t} \right) x^{-1} L_n(-x) \\ &= 2^{-n} x \sum_{k=0}^{n-1} C_n(k; 1) \frac{t^k}{k!} (x-1)^{-1} L_n(1-x) \\ &= 2^{-n} x \sum_{k=0}^{n-1} C_n(k; 1) \frac{1}{k!} \sum_{l=1}^n \binom{n-1}{l-1} \frac{n!}{l!} t^k (x-1)^{l-1} \\ &= \frac{n!}{2^n} \sum_{k=0}^{n-1} \sum_{l=1}^n \binom{n-1}{l-1} \binom{l-1}{k} \frac{C_n(k; 1)}{l!} x (x-1)^{l-1-k}. \end{aligned} \tag{2.24}$$

Thus, by (2.24), we obtain the following lemma.

Lemma 2.6. *For $n \in \mathbb{N}$, let $S_n(x) \sim \left(1, \frac{-2t}{1-t^2}\right)$. Then we have*

$$S_n(x) = \frac{n!}{2^n} \sum_{k=0}^{n-1} \sum_{l=1}^n \binom{n-1}{l-1} \binom{l-1}{k} \frac{C_n(k; 1)}{l!} x (x-1)^{l-1-k}.$$

As is known, we have

$$xB_{n-1}^{(an)}(x) \sim \left(1, t \left(\frac{e^t - 1}{t} \right)^a \right), \quad xE_{n-1}^{(bn)}(x) \sim \left(1, t \left(\frac{e^t + 1}{2} \right)^b \right), \tag{2.25}$$

where a, b are positive integers (see [10, 11, 17]). For $n \geq 1$, by (1.12) and (2.25), we get

$$\begin{aligned} xE_{n-1}^{(bn)}(x) &= x \left(\frac{t \left(\frac{e^t - 1}{t} \right)^a}{t \left(\frac{e^t + 1}{2} \right)^b} \right)^n x^{-1} B_{n-1}^{(an)}(x) \\ &= x \left(\frac{e^t + 1}{2} \right)^{-bn} \left(\frac{e^t - 1}{t} \right)^{an} B_{n-1}^{(an)}(x). \end{aligned} \tag{2.26}$$

Thus, by (2.26), we get

$$\left(\frac{e^t + 1}{2} \right)^{bn} E_{n-1}^{(bn)}(x) = \left(\frac{e^t - 1}{t} \right)^{an} B_{n-1}^{(an)}(x). \tag{2.27}$$

$$\begin{aligned}
\text{LHS of (2.27)} &= 2^{-bn} (e^t + 1)^{bn} E_{n-1}^{(bn)}(x) = 2^{-bn} \sum_{k=0}^{bn} \binom{bn}{k} e^{kt} E_{n-1}^{(bn)}(x) \\
&= 2^{-bn} \sum_{k=0}^{bn} \binom{bn}{k} E_{n-1}^{(bn)}(x+k).
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
\text{RHS of (2.27)} &= \left(\frac{1}{t}\right)^{an} (an)! \sum_{l=an}^{\infty} S_2(l, an) \frac{t^l}{l!} B_{n-1}^{(an)}(x) \\
&= \sum_{l=0}^{n-1} \frac{(an)!}{(l+an)!} S_2(l+an, an) (n-1)_l B_{n-1-l}^{(an)}(x) \\
&= (n-1)! \sum_{l=0}^{n-1} \frac{(an)!}{(l+an)!(n-1-l)!} S_2(l+an, an) B_{n-1-l}^{(an)}(x),
\end{aligned} \tag{2.29}$$

where $S_2(n, k)$ is the Stirling number of the second kind. Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.7. *For $n \geq 1$, $a, b \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{k=0}^{bn} \binom{bn}{k} E_{n-1}^{(bn)}(x+k) = 2^{bn} (n-1)! \sum_{l=0}^{n-1} \frac{(an)!}{(l+an)!(n-1-l)!} S_2(l+an, an) B_{n-1-l}^{(an)}(x).$$

The Pidduck sequences is given by

$$P_n(x) \sim \left(\frac{2}{e^t + 1}, \frac{e^t - 1}{e^t + 1} \right). \tag{2.30}$$

From (2.30), we can derive the generating function of the Pidduck sequences as follows:

$$\sum_{k=0}^{\infty} P_k(x) \frac{t^k}{k!} = (1-t)^{-1} \left(\frac{1+t}{1-t} \right)^x. \tag{2.31}$$

Let $S_n(x) \sim \left(1, \frac{2t}{e^t + 1} \right)$. Then, from (1.12), (1.15) and (2.30), we have

$$\begin{aligned}
M_n(x) &= \frac{2}{e^t + 1} P_n(x) = x \left(\frac{\frac{2t}{e^t + 1}}{\frac{e^t - 1}{e^t + 1}} \right)^n x^{-1} S_n(x) \\
&= 2^n x \left(\frac{t}{e^t - 1} \right)^n x^{-1} S_n(x).
\end{aligned} \tag{2.32}$$

By (1.12), we easily get

$$\begin{aligned}
S_n(x) &= x \left(\frac{\frac{2t}{e^t + 1}}{t} \right)^{-n} x^{-1} x^n = 2^{-n} x (e^t + 1)^n x^{n-1} \\
&= 2^{-n} x \sum_{j=0}^n \binom{n}{j} e^{jt} x^{n-1} = 2^{-n} x \sum_{j=0}^n \binom{n}{j} (x+j)^{n-1}.
\end{aligned} \tag{2.33}$$

From (2.32) and (2.33), we have

$$\begin{aligned}
 M_n(x) &= 2^n x \left(\frac{t}{e^t - 1} \right)^n \left(2^{-n} \sum_{j=0}^n \binom{n}{j} (x+j)^{n-1} \right) \\
 &= \sum_{j=0}^n \binom{n}{j} x \left(\frac{t}{e^t - 1} \right)^n (x+j)^{n-1} = \sum_{j=0}^n \binom{n}{j} x B_{n-1}^{(n)}(x+j).
 \end{aligned} \tag{2.34}$$

By (2.32) and (2.34), we get

$$\begin{aligned}
 P_n(x) &= \frac{1}{2}(e^t + 1)M_n(x) = \frac{1}{2}(e^t + 1) \sum_{j=0}^n \binom{n}{j} x B_{n-1}^{(n)}(x+j) \\
 &= \frac{1}{2} \sum_{j=0}^n \binom{n}{j} \left\{ (x+1)B_{n-1}^{(n)}(x+1+j) + xB_{n-1}^{(n)}(x+j) \right\} \\
 &= \frac{1}{2} \left\{ \sum_{j=1}^n \left(\binom{n}{j-1}(x+1) + \binom{n}{j}x \right) B_{n-1}^{(n)}(x+j) + (x+1)B_{n-1}^{(n)}(x+n+1) + xB_{n-1}^{(n)}(x) \right\} \\
 &= \frac{1}{2} \sum_{j=0}^{n+1} \left\{ \binom{n+1}{j}x + \binom{n}{j-1} \right\} B_{n-1}^{(n)}(x+j).
 \end{aligned} \tag{2.35}$$

Therefore, by (2.35), we obtain the following theorem.

Theorem 2.8. *For $n \geq 1$, we have*

$$P_n(x) = \frac{1}{2} \sum_{j=0}^{n+1} \left\{ \binom{n+1}{j}x + \binom{n}{j-1} \right\} B_{n-1}^{(n)}(x+j).$$

Let us consider the following two associated sequences:

$$S_n(x) \sim \left(1, \frac{2t}{e^t + 1} \right), \quad M_n(x) \sim \left(1, \frac{e^t - 1}{e^t + 1} \right). \tag{2.36}$$

For $n \geq 1$, by (1.12), we get

$$\begin{aligned}
 S_n(x) &= x \left(\frac{\frac{e^t - 1}{e^t + 1}}{\frac{2t}{e^t + 1}} \right)^n x^{-1} M_n(x) \\
 &= 2^{-n} x \left(\frac{e^t - 1}{t} \right)^n x^{-1} M_n(x).
 \end{aligned} \tag{2.37}$$

By (2.34) and (2.37), we get

$$\begin{aligned}
S_n(x) &= 2^{-n} x \frac{1}{t^n} n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!} x^{-1} M_n(x) \\
&= 2^{-n} x \sum_{l=0}^{n-1} \frac{n! S_2(l+n, n)}{(l+n)!} t^l (x^{-1} M_n(x)) \\
&= 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) \sum_{j=0}^n \binom{n}{j} (n-1)_l B_{n-1-l}^{(n)}(x+j) \\
&= 2^{-n} x n! (n-1)! \sum_{l=0}^{n-1} \sum_{j=0}^n \frac{\binom{n}{j} S_2(l+n, n)}{(l+n)! (n-l-1)!} B_{n-1-l}^{(n)}(x+j).
\end{aligned} \tag{2.38}$$

Therefore, by (2.33) and (2.36), we obtain the following theorem.

Theorem 2.9. *For $n \geq 1$, we have*

$$\sum_{j=0}^n \binom{n}{j} (x+j)^{n-1} = n! (n-1)! \sum_{l=0}^{n-1} \sum_{j=0}^n \frac{\binom{n}{j} S_2(l+n, n)}{(l+n)! (n-l-1)!} B_{n-1-l}^{(n)}(x+j).$$

Moreover,

$$\sum_{j=0}^n \binom{n}{j} j^{n-1} = n! (n-1)! \sum_{l=0}^{n-1} \sum_{j=0}^n \frac{\binom{n}{j} S_2(l+n, n)}{(l+n)! (n-l-1)!} B_{n-1-l}^{(n)}(j).$$

By (1.15), we get

$$x^{-1} M_n(x) = \sum_{k=1}^n \binom{n}{k} (n-1)_{n-k} 2^k \sum_{j=0}^{k-1} S_1(k-1, j) (x-1)^j, \tag{2.39}$$

where $S_1(k, j)$ is the Stirling number of the first kind. From (2.38) and (2.39), we can derive

$$\begin{aligned}
S_n(x) &= 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) t^l (x^{-1} M_n(x)) \\
&= 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) t^l \sum_{k=1}^n \binom{n}{k} (n-1)_{n-k} 2^k \sum_{j=0}^{k-1} S_1(k-1, j) (x-1)^j \\
&= 2^{-n} x n! (n-1)! \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \sum_{k=j+1}^n \frac{\binom{n}{k} 2^k S_2(l+n, n) S_1(k-1, j)}{(l+n)! (k-1)!} t^l (x-1)^j \\
&= 2^{-n} x n! (n-1)! \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{k=j+1}^n \frac{\binom{n}{k} 2^k S_2(l+n, n) S_1(k-1, j)}{(l+n)! (k-1)!} (j)_l (x-1)^{j-l}.
\end{aligned} \tag{2.40}$$

Therefore, by (2.33) and (2.40), we obtain the following theorem.

Theorem 2.10. *For $n \geq 1$, we have*

$$\sum_{j=0}^n \binom{n}{j} (x+j)^{n-1} = n! (n-1)! \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{k=j+1}^n \frac{\binom{n}{k} 2^k S_2(l+n, n) S_1(k-1, j) j!}{(l+n)! (k-1)! (j-l)!} (x-1)^{j-l}.$$

REMARK. From (2.34), we note that

$$x^{-1}M_n(x) = \sum_{k=0}^n \binom{n}{k} (x+k-1)_{n-1}. \quad (2.41)$$

By (2.38) and (2.41), we get

$$\begin{aligned} S_n(x) &= 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) t^l (x^{-1}M_n(x)) \\ &= 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) t^l \left(\sum_{k=0}^n \binom{n}{k} (x+k-1)_{n-1} \right) \\ &= 2^{-n} x n! \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{k=0}^n \frac{\binom{n}{l} S_2(l+n, n) S_1(n-1, j)}{(l+n)!} (j)_l (x+k-1)^{j-l} \\ &= 2^{-n} x n! \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{k=0}^n \frac{\binom{n}{k} S_2(l+n, n) S_1(n-1, j) j!}{(l+n)! (j-l)!} (x+k-1)^{j-l}. \end{aligned} \quad (2.42)$$

So, by (2.33) and (2.42), we get

$$\sum_{j=0}^n \binom{n}{j} (x+j)^{n-1} = n! \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{k=0}^n \frac{\binom{n}{k} S_2(l+n, n) S_1(n-1, j) j!}{(l+n)! (j-l)!} (x+k-1)^{j-l}. \quad (2.43)$$

The Narumi polynomials $N_n^{(a)}(x)$ of order a is defined by the generating function to be

$$\sum_{k=0}^{\infty} \frac{N_k^{(a)}(x)}{k!} t^k = \left(\frac{\log(1+t)}{t} \right)^a (1+t)^x. \quad (2.44)$$

Thus, from (2.44), we see that

$$N_n^{(a)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^a, e^t - 1 \right), \text{ (see [17,18])}. \quad (2.45)$$

In the special case, $x = 0$, $N_k^{(a)}(0) = N_k^{(a)}$ are called the k -th Narumi numbers of order a . If $a = 1$ in (2.45), then we will write $N_n(x)$ and N_n for $N_n^{(1)}(x)$ and $N_n^{(1)}$.

By (1.12) and (1.16), we get

$$\begin{aligned} \phi_n(x) &= x \left(\frac{t}{\log(1+t)} \right)^n x^{-1} x^n = x \left(\frac{t}{\log(1+t)} \right)^n x^{n-1} \\ &= x \left(\sum_{k=0}^{\infty} \frac{N_k^{(-n)}(0)}{k!} t^k \right) x^{n-1} = x \sum_{k=0}^{n-1} \frac{N_k^{(-n)}}{k!} (n-1)_k x^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} N_k^{(-n)} x^{n-k} = \sum_{k=1}^n \binom{n-1}{k-1} N_{n-k}^{(-n)} x^k. \end{aligned} \quad (2.46)$$

Therefore, by (1.16) and (2.46), we obtain the following lemma.

Lemma 2.11. For $n, k \in \mathbb{N}$ with $k \leq n$, we have

$$S_2(n, k) = \binom{n-1}{k-1} N_{n-k}^{(-n)}.$$

By (1.12), (1.16) and (1.17), we get

$$\begin{aligned}
\phi_n(x) &= x \left(\frac{t}{(1+t)\log(1+t)} \right)^n x^{-1} L_n(-x) \\
&= x \left(\frac{t}{\log(1+t)} \right)^n (1+t)^{-n} \sum_{l=1}^n \binom{n-1}{l-1} \frac{n!}{l!} x^{l-1} \\
&= x \left(\sum_{k=0}^{\infty} \frac{N_k^{(-n)}(-n)}{k!} t^k \right) \sum_{l=1}^n \binom{n-1}{l-1} \frac{n!}{l!} x^{l-1} \\
&= x \sum_{k=0}^{n-1} \frac{N_k^{(-n)}(-n)}{k!} \sum_{l=k+1}^n \binom{n-1}{l-1} \frac{n!}{l!} (l-1)_k x^{l-1-k} \\
&= n! \sum_{k=0}^{n-1} \sum_{l=k+1}^n \frac{\binom{n-1}{l-1} \binom{l-1}{k}}{l!} N_k^{(-n)}(-n) x^{l-k} \\
&= n! \sum_{k=0}^{n-1} \sum_{m=1}^{n-k} \frac{\binom{n-1}{k+m-1} \binom{k+m-1}{k}}{(k+m)!} N_k^{(-n)}(-n) x^m \\
&= n! \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \frac{\binom{n-1}{k+m-1} \binom{k+m-1}{k}}{(k+m)!} N_k^{(-n)}(-n) \right\} x^m.
\end{aligned} \tag{2.47}$$

From (1.16) and (2.47), we have

$$S_2(n, m) = n! \sum_{k=0}^{n-m} \frac{\binom{n-1}{k+m-1} \binom{k+m-1}{k}}{(k+m)!} N_k^{(-n)}(-n), \tag{2.48}$$

where $1 \leq m \leq n$.

Therefore, by Lemma 2.11 and (2.48), we obtain the following theorem.

Theorem 2.12. *For $m, n \in \mathbb{N}$ with $m \leq n$, we have*

$$\binom{n-1}{m-1} N_{n-m}^{(-n)} = n! \sum_{k=0}^{n-m} \frac{\binom{n-1}{k+m-1} \binom{k+m-1}{k}}{(k+m)!} N_k^{(-n)}(-n).$$

It is well known that

$$\left(\frac{t}{\log(1+t)} \right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!}, \quad (\text{see [17]}). \tag{2.49}$$

Thus, by (2.44) and (2.49), we get

$$\sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!} = \left(\frac{t}{\log(1+t)} \right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} N_k^{(-n)}(x-1) \frac{t^k}{k!}. \tag{2.50}$$

By comparing the coefficients on the both sides of (2.50), we get

$$B_k^{(k-n+1)}(x) = N_k^{(-n)}(x-1). \tag{2.51}$$

Therefore, by (2.51), we obtain the following corollary.

Corollary 2.13. *For $m, n \in \mathbb{N}$ with $m \leq n$, we have*

$$\binom{n-1}{m-1} B_{n-m}^{(-m+1)}(1) = n! \sum_{k=0}^{n-m} \left(\frac{\binom{n-1}{k+m-1} \binom{k+m-1}{k}}{(k+m)!} B_k^{(k-n+1)}(-n+1) \right).$$

Let us consider the following associated sequence:

$$S_n(x) \sim (1, t(1+t)^a), \quad a \neq 0. \quad (2.52)$$

Then, by (1.12) and (2.52), we get

$$\begin{aligned} S_n(x) &= x \left(\frac{t}{t(1+t)^a} \right)^n x^{-1} x^n = x(1+t)^{-an} x^{n-1} \\ &= x \sum_{l=0}^{n-1} \binom{-an}{l} t^l x^{n-1} = \sum_{l=0}^{n-1} \binom{-an}{l} (n-1)_l x^{n-l} \\ &= \sum_{l=1}^n \binom{-an}{n-l} (n-1)_{n-l} x^l. \end{aligned} \quad (2.53)$$

For $n \geq 1$, from (1.16) and (2.52), we have

$$\begin{aligned} \phi_n(x) &= x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n x^{-1} S_n(x) \\ &= x \left(\frac{t}{\log(1+t)} \right)^n (1+t)^{an} x^{-1} S_n(x) \\ &= x \left(\sum_{k=0}^{n-1} \frac{N_k^{(-n)}(an)}{k!} t^k \right) x^{-1} S_n(x). \end{aligned} \quad (2.54)$$

By (2.53) and (2.54), we get

$$\begin{aligned} \phi_n(x) &= x \sum_{k=0}^{n-1} \frac{N_k^{(-n)}(an)}{k!} \sum_{l=1}^n \binom{-an}{n-l} (n-1)_{n-l} t^k x^{l-1} \\ &= x \sum_{k=0}^{n-1} \frac{N_k^{(-n)}(an)}{k!} \sum_{l=k+1}^n \binom{-an}{n-l} (n-1)_{n-l} (l-1)_k x^{l-1-k} \\ &= (n-1)! \sum_{k=0}^{n-1} \sum_{l=k+1}^n \frac{\binom{-an}{n-l} \binom{l-1}{k}}{(l-1)!} N_k^{(-n)}(an) x^{l-k} \\ &= (n-1)! \sum_{k=0}^{n-1} \sum_{m=1}^{n-k} \frac{\binom{-an}{n-k-m} \binom{k+m-1}{k}}{(k+m-1)!} N_k^{(-n)}(an) x^m \\ &= (n-1)! \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \frac{\binom{-an}{n-k-m} \binom{k+m-1}{k}}{(k+m-1)!} N_k^{(-n)}(an) \right\} x^m \\ &= (n-1)! \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \frac{\binom{-an}{n-k-m} \binom{k+m-1}{k}}{(k+m-1)!} B_k^{(k-n+1)}(an+1) \right\} x^m. \end{aligned} \quad (2.55)$$

Therefore, by (1.16) and (2.55), we obtain the following theorem.

Theorem 2.14. *For $m, n \in \mathbb{N}$ with $m \leq n$, we have*

$$\binom{n-1}{m-1} B_{n-m}^{(-m+1)}(1) = (n-1)! \sum_{k=0}^{n-m} \frac{\binom{-an}{n-k-m} \binom{k+m-1}{k}}{(k+m-1)!} B_k^{(k-n+1)}(an+1).$$

REMARKS (I). For $n \geq 1$, we have

$$\begin{aligned} x^n &= x \left(\frac{\log(1+t)}{t} \right)^n x^{-1} \phi_n(x) \\ &= \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)} \right\} x^m. \end{aligned} \quad (2.56)$$

By comparing the coefficients on the both sides of (2.56), we get

$$\sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)} = \delta_{m,n}, \quad (2.57)$$

where $1 \leq m \leq n$.

(II). For $n \geq 1$, we have

$$\begin{aligned} L_n(-x) &= x \left(\frac{\log(1+t)}{\frac{t}{1+t}} \right)^n x^{-1} \phi_n(x) \\ &= \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)}(n) \right\} x^m. \end{aligned} \quad (2.58)$$

By (1.17) and (2.58), we get

$$\binom{n-1}{m-1} \frac{n!}{m!} = \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)}(n), \quad (2.59)$$

where $1 \leq m \leq n$.

As is known, the Laguerre polynomials of order α are given by Sheffer sequences to be

$$L_n^{(\alpha)}(x) \sim \left((1-t)^{-\alpha-1}, \frac{t}{t-1} \right). \quad (2.60)$$

Thus, by the definition of Sheffer sequence, we get

$$\left\langle (1+t)^{-\alpha-1} \left(\frac{t}{t+1} \right)^k \middle| L_n^{(\alpha)}(-x) \right\rangle = n! \delta_{n,k} \quad (n, k \geq 0). \quad (2.61)$$

By (2.61), we easily see that

$$L_n(-x) = (1+t)^{-\alpha-1} L_n^{(\alpha)}(-x) \sim \left(1, \frac{t}{1+t} \right). \quad (2.62)$$

From (2.58) and (2.62), we have

$$\begin{aligned} (1+t)^{-\alpha-1} L_n^{(\alpha)}(-x) &= L_n(-x) \\ &= \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)}(n) \right\} x^m. \end{aligned} \quad (2.63)$$

Thus, by (2.63), we get

$$\begin{aligned} L_n^{(\alpha)}(-x) &= (1+t)^{\alpha+1} \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)}(n) \right\} x^m \\ &= \sum_{l=0}^n \left\{ \sum_{m=l}^n \sum_{k=0}^{n-m} \binom{k+m-1}{k} \binom{\alpha+1}{m-l} (m)_{m-l} S_2(n, k+m) N_k^{(n)}(n) \right\} x^l. \end{aligned} \quad (2.64)$$

It is known that

$$L_n^{(\alpha)}(-x) = \sum_{l=0}^n \binom{n+\alpha}{n-l} \frac{n!}{l!} x^l. \quad (2.65)$$

By (2.64) and (2.65), we get

$$\binom{n+\alpha}{n-l} \frac{n!}{l!} = \sum_{m=l}^n \sum_{k=0}^{n-m} \binom{k+m-1}{k} \binom{\alpha+1}{m-l} (m)_{m-l} S_2(n, k+m) N_k^{(n)}(n),$$

where $0 \leq l \leq n$.

Finally, we consider the following associated sequences:

$$S_n(x) = \sum_{k=1}^n \binom{-an}{n-k} (n-1)_{n-k} x^k \sim (1, t(1+t)^a), \quad a \neq 0. \quad (2.66)$$

Thus, by (1.12) and (2.66), we get

$$\begin{aligned} S_n(x) &= x \left(\frac{\log(1+t)}{t(1+t)^a} \right)^n x^{-1} \phi_n(x) \\ &= x \left(\frac{\log(1+t)}{t} \right)^n (1+t)^{-an} x^{-1} \phi_n(x) \\ &= x \sum_{k=0}^{n-1} \frac{N_k^{(n)}(-an)}{k!} t^k \sum_{l=0}^n S_2(n, l) x^{l-1} \\ &= \sum_{m=1}^n \left\{ \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)}(-an) \right\} x^m. \end{aligned} \quad (2.67)$$

From (2.66) and (2.67), we have

$$\binom{-an}{n-m} (n-1)_{n-m} = \sum_{k=0}^{n-m} \binom{k+m-1}{k} S_2(n, k+m) N_k^{(n)}(-an),$$

where $m, n \in \mathbb{N}$ with $m \leq n$ and $a \neq 0$.

REFERENCES

- [1] L. Carlitz, *Some generating functions for Laguerre polynomials*, Duke Math. J., **35** (1968) 825-827.
- [2] L. Carlitz, *A note on the Bessel polynomials*, Duke Math. J., **24** (1957) 151-162.
- [3] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. **22** (2012) 433-438.
- [4] R. Dere and Y. Simsek, *Genocchi polynomials associated with the umbral algebra*, Appl. Math. Comput., **218** no. 3, 756-761.
- [5] B. Diarra, *Ultrametric umbral calculus in characteristic p*, Bull. Belg. Math. Soc. Simon Stevin, **14** (2007) 845-869.

- [6] T. Ernst, *Examples of a q -umbral calculus*, Adv. Stud. Contemp. Math. **16** no. 1, (2008) 1-22.
- [7] Q. Fang and T. Wang, *Umbral calculus and invariant sequences*, Ars Combin., **101** (2011), 257-264.
- [8] D. S. Kim, T. Kim, S.-H. Lee and S.-H. Rim, *Frobenius-Euler polynomials and umbral calculus in the p -adic case*, Adv. Difference Equ. 2012, **2012**:222.
- [9] D. S. Kim and T. Kim, *Some new identities of Frobenius-Euler numbers and polynomials*, J. of Ineq. and Appl., 2012, **2012**:307.
- [10] D. S. Kim and T. Kim, *Applications of Umbral Calculus Associated with p -Adic Invariant Integrals on \mathbb{Z}_p* , Abstract and Applied Analysis 2012 (2012), Article ID 865721, 12 pages.
- [11] D. S. Kim, T. Kim, S.-H. Lee and S.-H. Rim, *Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Adv. Difference Equ. 2012, **2012**:196.
- [12] D. S. Kim, T. Kim, S.-H. Lee and Y.-H. Kim, *Some identities for the product of two Bernoulli and Euler polynomials*, Adv. Difference Equ. 2012, **2012**:95.
- [13] T. Kim, S.-H. Rim, D. V. Dolgy and S.-H. Lee, *Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials*, Adv. Difference Equ. 2012, **2012**:201.
- [14] T. Kim, *Some identities on the q -Euler polynomials of higher order and q -Stirling numbers by the fermionic p -adic integral on \mathbb{Z}_p* , Russ. J. Phys. **16** (2009) no. 4, 484-491.
- [15] A. K. Kwaśniewski, *More on the Bernoulli-Taylor formula for extended umbral calculus*, Adv. Appl. Clifford Algebr., **16** (2006), no. 1, 29-39.
- [16] T. J. Robinson, *Formal calculus and umbral calculus*, Electron. J. Combin., **17** (2010), no. 1, Research Paper 95, 31 pp.
- [17] S. Roman, *More on the umbral calculus, with emphasis on the q -umbral calculus*, J. Math. Anal. Appl., **107** (1985), 222-254.
- [18] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.
- [19] C. Ryoo, *Some relations between twisted q -Euler numbers and Bernstein polynomials*, Adv. Stud. Contemp. Math., **21** (2011), no. 2, 271-223.
- [20] Y. Simsek, *Special functions related to Dedekind-type DC -sums and their applications*, Russ. J. Math. Phys. **17** (2010) 495-508.
- [21] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*, Adv. Stud. Contemp. Math., **16** (2008), no. 2, 251-278 .
- [22] X.-H. Sun, *On umbral calculus, I*, J. Math. Anal. Appl., **224** (2000), no. 2, 279-290.

¹ DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA.

E-mail address: `tkkim@kw.ac.kr`

² DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA.

E-mail address: `dskim@sogang.ac.kr`